Chapter 3
Interval Estimation and Hypothesis Testing

Walter R. Paczkowski
Rutgers University

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3.1 Interval Estimation

- There are two types of estimates
  - Point estimates
    - The estimate b_2 is a point estimate of the unknown population parameter in the regression model.
  - Interval estimates
    - Interval estimation proposes a range of values in which the true parameter is likely to fall
    - Providing a range of values gives a sense of what the parameter value might be, and the precision with which we have estimated it
    - Such intervals are often called **confidence intervals**.
      - We prefer to call them **interval estimates** because the term "confidence" is widely misunderstood and misused

- The normal distribution of b_2, the least squares estimator of β_2, is

$$b_2 \sim N\left(\beta_2, \frac{\sigma^2}{\sum (x_i - \bar{x})^2}\right)$$

- A standardized normal random variable is obtained from b_2 by subtracting its mean and dividing by its standard deviation:

Eq. 3.1

$$Z = \frac{b2 - \beta_2}{\sqrt{\sigma^2 / \sum (x_i - \bar{x})^2}} \sim N(0,1)$$

- We know that:

$$P(-1.96 \le Z \le 1.96) = 0.95$$

- Substituting:

$$P\left(-1.96 \le \frac{b2 - \beta_2}{\sqrt{\sigma^2 / \sum (x_i - \bar{x})^2}} \le 1.96 \right) = 0.95$$

- Rearranging:

$$P\left(b_2 - 1.96\sqrt{\sigma^2 / \sum (x_i - \bar{x})^2} \le \beta_2 \le b_2 + 1.96\sqrt{\sigma^2 / \sum (x_i - \bar{x})^2} \right) = 0.95$$

- The two end-points $b_2 \pm 1.96\sqrt{\sigma^2 / \sum (x_i - \bar{x})^2}$ provide an interval estimator.
- In repeated sampling 95% of the intervals constructed this way will contain the true value of the parameter β_2.
- This easy derivation of an interval estimator is based on the assumption SR6 *and* that we know the variance of the error term σ^2.

- Replacing σ^2 with $\hat{\sigma}^2$ creates a random variable *t*:

Eq. 3.2

$$t = \frac{b_2 - \beta_2}{\sqrt{\sigma^2 / \sum (x_i - \bar{x})^2}} = \frac{b_2 - \beta_2}{\sqrt{\hat{\text{var}}(b_2)}} = \frac{b_2 - \beta_2}{se(b_2)} \sim t_{(N-2)}$$

- The ratio $t = b_2 - \beta_2 / se(b_2)$ has a *t*-distribution with (N – 2) degrees of freedom, which we denote as:

$$t \sim t_{(N-2)}$$

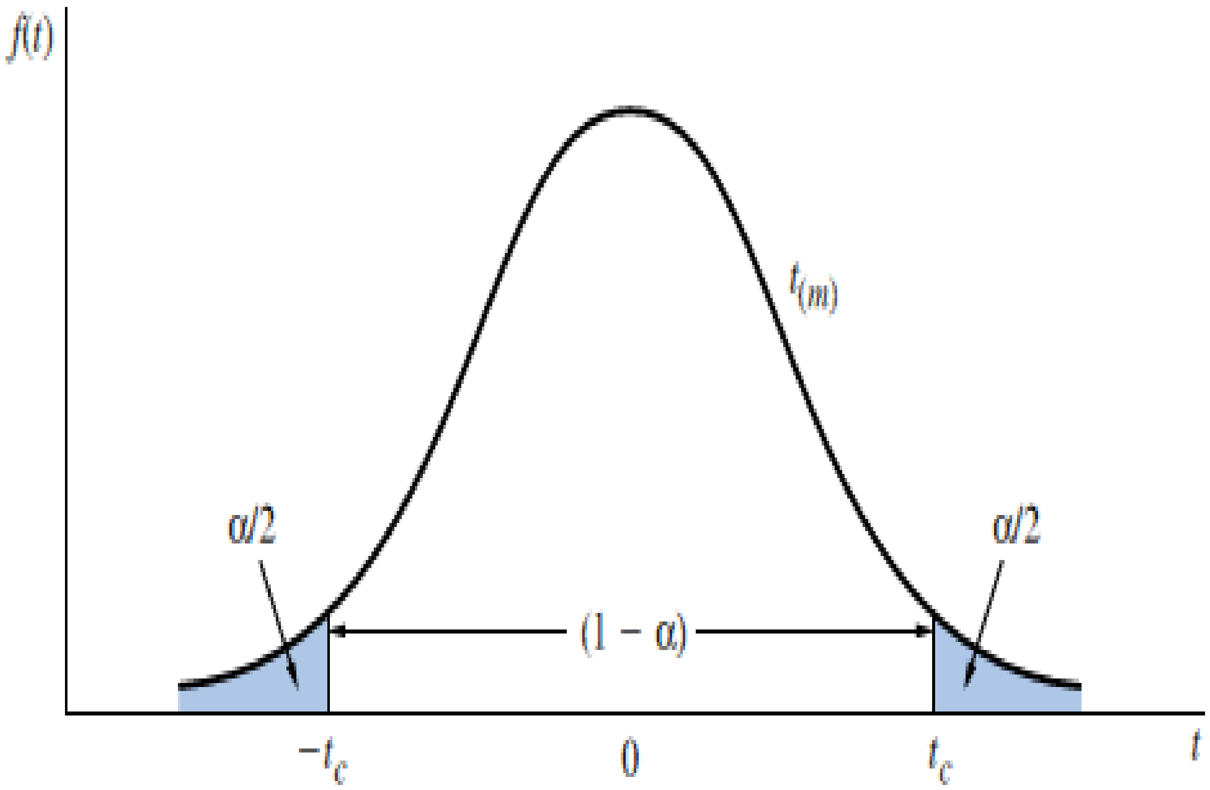
- In general we can say, if assumptions SR1-SR6 hold in the simple linear regression model, then

Eq. 3.3

$$t = \frac{b_k - \beta_k}{se(b_k)} \sim t_{(N-2)} \text{ for } k = 1,2$$

- The *t*-distribution is a bell shaped curve centered at zero
- It looks like the standard normal distribution, except it is more spread out, with a larger variance and thicker tails
- The shape of the *t*-distribution is controlled by a single parameter called the **degrees of freedom**, often abbreviated as *df*

Figure 3.1 Critical values from a *t*-distribution.



- We can find a "critical value" from a t-distribution such that

$$P(t \geq t_c) = P(t \leq -t_c) = \alpha/2$$

where α is a probability often taken to be α = 0.01 or α = 0.05.

- The critical value t_c for degrees of freedom *m* is the percentile value $t_{(1-\alpha/2,\ m)}$

- Each shaded "tail" area contains α/2 of the probability, so that 1-α of the probability is contained in the center portion.
  - Consequently, we can make the probability statement

Eq. 3.4

$$P(-t_c \le t \le t_c) = 1 - \alpha$$

or

$$P\left(-t_c \le \frac{b_k - \beta_k}{se(b_k)} \le t_c\right) = 1 - \alpha$$

or

Eq. 3.5

$$P[b_k - t_c se(b_k) \le \beta_k \le t_c + t_c se(b_k)] = 1 - \alpha$$

- When b_k and $se(b_k)$ are estimated values (numbers), based on a given sample of data, then $b_k \pm t_c se(b_k)$ is called a $100(1-\alpha)\%$ interval estimate of b_k.
  - Equivalently it is called a $100(1-\alpha)\%$ confidence interval.
  - Usually $\alpha = 0.01$ or $\alpha = 0.05$, so that we obtain a 99% confidence interval or a 95% confidence interval.

- The interpretation of confidence intervals requires a great deal of care
  - The properties of the interval estimation procedure are based on the notion of repeated sampling
  - Any one interval estimate, based on one sample of data, may or may not contain the true parameter β_k, and because β_k is unknown, we will never know whether it does or does not
  - When "confidence intervals" are discussed, remember that our confidence is in the procedure used to construct the interval estimate; it is not in any one interval estimate calculated from a sample of data

- For the food expenditure data

Eq. 3.6

$$P\left[b_2 - 2.024se(b_2) \le \beta_2 \le b_2 + 2.024se(b_2)\right] = 0.95$$

– The critical value t_c = 2.024, which is appropriate for α = .05 and 38 degrees of freedom

– To construct an interval estimate for β_2 we use the least squares estimate b_2 = 10.21 and its standard error

$$se(b_2) = \sqrt{\widehat{\text{var}}(b_2)} = \sqrt{4.38} = 2.09$$

- A "95% confidence interval estimate" for β_2:

$$b_2 \pm t_c se(b_2) = 10.21 \pm 2.024(2.09) = [5.97, 14.45]$$

  – When the procedure we used is applied to many random samples of data from the same population, then 95% of all the interval estimates constructed using this procedure will contain the true parameter

- Is β_2 actually in the interval [5.97, 14.45]?
  - We do not know, and we will never know
  - What we do know is that when the procedure we used is applied to many random samples of data from the same population, then 95% of all the interval estimates constructed using this procedure will contain the true parameter
  - The interval estimation procedure “works” 95% of the time
    - What we can say about the interval estimate based on our one sample is that, given the reliability of the procedure, we would be “surprised” if β_2 is not in the interval [5.97, 14.45].

- What is the usefulness of an interval estimate of β_2?
  - When reporting regression results we always give a point estimate, such as $b_2 = 10.21$
  - However, the point estimate alone gives no sense of its reliability
  - Thus, we might also report an interval estimate
    - Interval estimates incorporate both the point estimate and the standard error of the estimate, which is a measure of the variability of the least squares estimator

Table 3.1 Least Squares Estimates from 10 Random Samples

Sample	b_1	$se(b_1)$	b_2	$se(b_2)$	$\hat{\sigma}^2$
1	131.69	40.58	6.48	1.96	7002.85
2	57.25	33.13	10.88	1.60	4668.63
3	103.91	37.22	8.14	1.79	5891.75
4	46.50	33.33	11.90	1.61	4722.58
5	84.23	41.15	9.29	1.98	7200.16
6	26.63	45.78	13.55	2.21	8911.43
7	64.21	32.03	10.93	1.54	4362.12
8	79.66	29.87	9.76	1.44	3793.83
9	97.30	29.14	8.05	1.41	3610.20
10	95.96	37.18	7.77	1.79	5878.71

Table 3.2 Interval Estimates from 10 Random Samples

Sample	$b_1 - t_c\text{se}(b_1)$	$b_1 + t_c\text{se}(b_1)$	$b_2 - t_c\text{se}(b_2)$	$b_2 + t_c\text{se}(b_2)$
1	49.54	213.85	2.52	10.44
2	−9.83	124.32	7.65	14.12
3	28.56	179.26	4.51	11.77
4	−20.96	113.97	8.65	15.15
5	0.93	167.53	5.27	13.30
6	−66.04	119.30	9.08	18.02
7	−0.63	129.05	7.81	14.06
8	19.19	140.13	6.85	12.68
9	38.32	156.29	5.21	10.89
10	20.69	171.23	4.14	11.40

3.2
Hypothesis Tests

- Hypothesis testing procedures compare a conjecture we have about a population to the information contained in a sample of data
  - Given an economic and statistical model, hypotheses are formed about economic behavior.
  - These hypotheses are then represented as statements about model parameters
  - Hypothesis tests use the information about a parameter that is contained in a sample of data, its least squares point estimate, and its standard error, to draw a conclusion about the hypothesis

COMPONENTS OF HYPOTHESIS TESTS

1. A null hypothesis H_0
2. An alternative hypothesis H_1
3. A test statistic
4. A rejection region
5. A conclusion

- A null hypothesis is the belief we will maintain until we are convinced by the sample evidence that it is not true, in which case we reject the null hypothesis
  - The null hypothesis is stated as $H_0: \beta_k = c$, where c is a constant, and is an important value in the context of a specific regression model
    - A common value for c is 0

- Paired with every null hypothesis is a logical alternative hypothesis H_1 that we will accept if the null hypothesis is rejected
  - The alternative hypothesis is flexible and depends to some extent on economic theory

- Possible Alternative hypotheses are:

$$H_1: \beta_k > c$$

$$H_1: \beta_k < c$$

$$H_1: \beta_k \neq c$$

- Based on the value of a test statistic we decide either to reject the null hypothesis or not to reject it
  - A test statistic has a special characteristic: its probability distribution is completely known when the null hypothesis is true, and it has some other distribution if the null hypothesis is not true

- The primary test statistic is:

$$t = \frac{b_k - \beta_k}{\mathrm{se}(b_k)} \sim t_{(N-2)}$$

  – If the null hypothesis H_0:b_k= c is true, then we can substitute c for b_k and it follows that:

Eq. 3.7

$$t = \frac{b_k - c}{\mathrm{se}(b_k)} \sim t_{(N-2)}$$

  – If the null hypothesis is not true, then the t-statistic in Eq. 3.7 does not have a *t*-distribution with *N*-2 degrees of freedom

- The rejection region depends on the form of the alternative
  - It is the range of values of the test statistic that leads to rejection of the null hypothesis
  - It is possible to construct a rejection region only if we have:
    - A test statistic whose distribution is known when the null hypothesis is true
    - An alternative hypothesis
    - A level of significance

- The rejection region consists of values that are unlikely and that have low probability of occurring when the null hypothesis is true
  - The chain of logic is:

    "*If a value of the test statistic is obtained that falls in a region of low probability, then it is unlikely that the test statistic has the assumed distribution, and thus it is unlikely that the null hypothesis is true*"

- If the alternative hypothesis is true, then values of the test statistic will tend to be unusually large or unusually small
  - The terms "large" and "small" are determined by choosing a probability α, called the **level of significance of the test**, which provides a meaning for "an unlikely event"
  - The level of significance of the test α is usually chosen to be 0.01, 0.05 or 0.10

- We could commit two types of error
  - If we reject the null hypothesis when it is true, then we commit what is called a Type error
    - The level of significance of a test is the probability of committing a Type I error
    - P(Type I error) = α
  - If we do not reject a null hypothesis that is false, then we have committed a Type II error

- Do you reject the null hypothesis, or do you not reject the null hypothesis?
  - Avoid saying that you "accept" the null hypothesis
  - Make it standard practice to say what the conclusion means in the economic context of the problem you are working on and the economic significance of the finding

3.3

Rejection Regions for Specific Alternatives

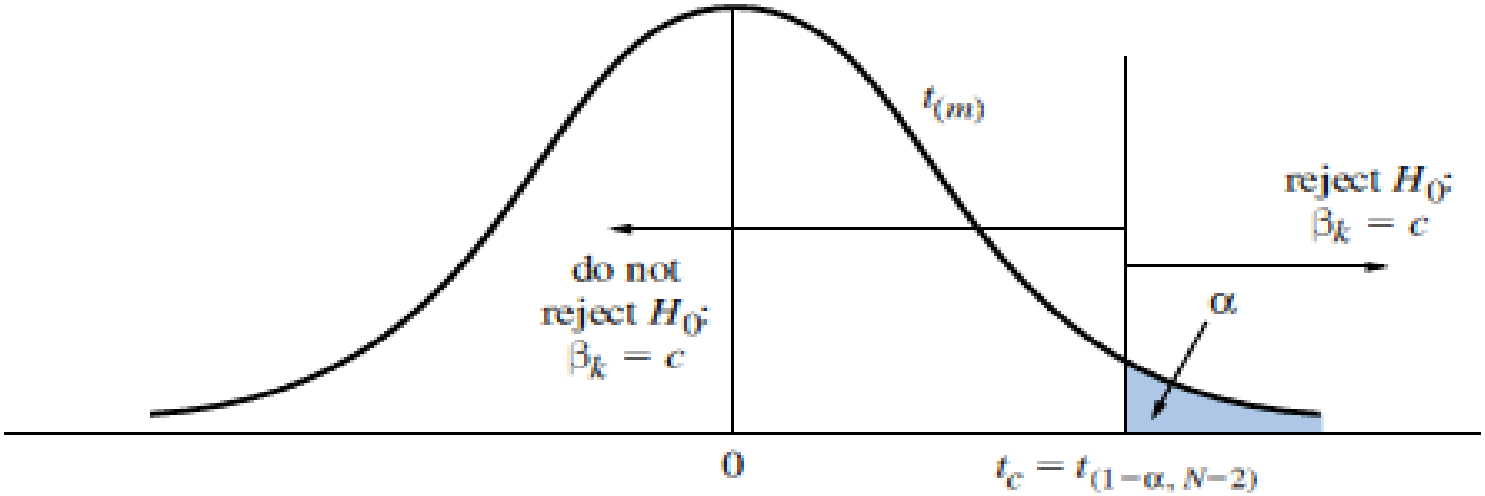
- To have a rejection region for a null hypothesis, we need:
  1. A test statistic
  2. A specific alternative
  3. A level of significance, α, for the test

3.3.1
One-tail Test with Alternative "Greater Than"

When testing the null hypothesis $H_0: \beta_k = c$ against the alternative hypothesis $H_1: \beta_k > c$, reject the null hypothesis and accept the alternative hypothesis if

$$t \geq t_{(1-\alpha; N-2)}$$

Figure 3.2 Rejection region for a one-tail test of H_0:$\beta_k = c$ against H_1:$\beta_k > c$

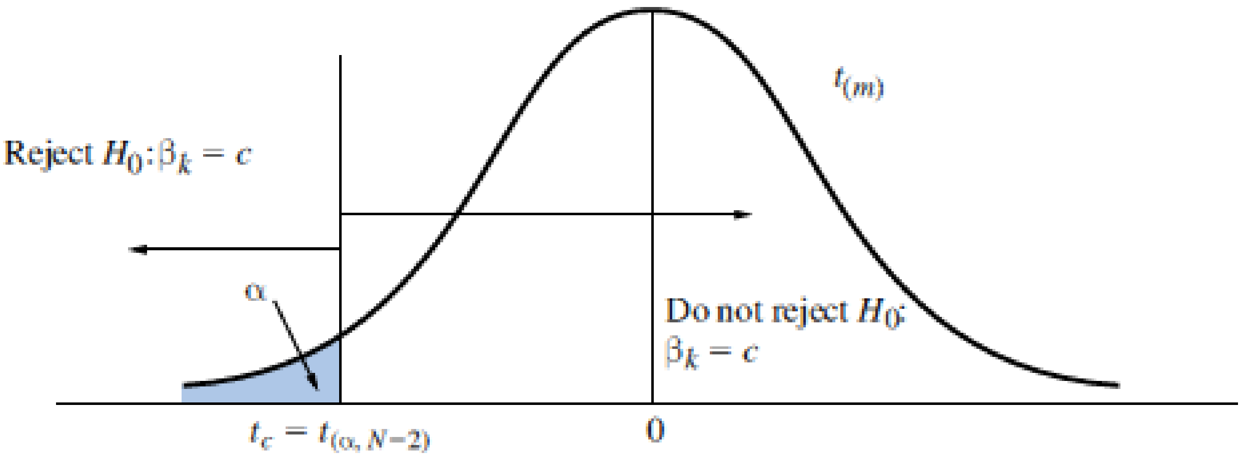


3.3.2
One-tail Test with Alternative "Less Than"

When testing the null hypothesis $H_0{:}\beta_k = c$ against the alternative hypothesis $H_1{:}\beta_k < c$, reject the null hypothesis and accept the alternative hypothesis if

$$t \leq t_{(1-\alpha;N-2)}$$

Figure 3.3 Rejection region for a one-tail test of H_0:$\beta_k = c$ against H_1:$\beta_k < c$

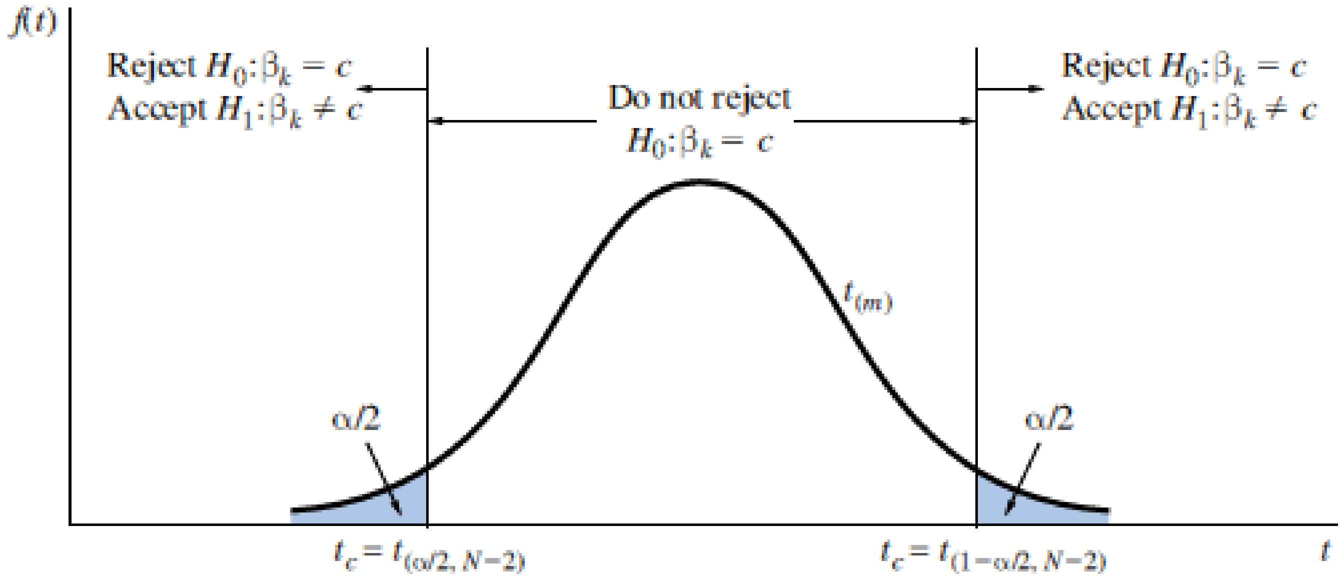


3.3.3
One-tail Test with Alternative "Not Equal To"

When testing the null hypothesis $H_0\!:\!\beta_k = c$ against the alternative hypothesis $H_1\!:\!\beta_k \neq c$, reject the null hypothesis and accept the alternative hypothesis if

$$t \leq t_{(1-\alpha;N-2)} \text{ or } t \geq t_{(1-\alpha;N-2)}$$

Figure 3.4 Rejection region for a test of $H_0{:}\beta_k = c$ against $H_1{:}\beta_k \neq c$



3.4
Examples of Hypothesis Tests

STEP-BY-STEP PROCEDURE FOR TESTING HYPOTHESES

1. Determine the null and alternative hypotheses.
2. Specify the test statistic and its distribution if the null hypothesis is true.
3. Select α and determine the rejection region.
4. Calculate the sample value of the test statistic.
5. State your conclusion.

- The null hypothesis is $H_0\!:\!\beta_2 = 0$
  The alternative hypothesis is $H_1\!:\!\beta_2 > 0$
- The test statistic is Eq. 3.7
  – In this case $c = 0$, so $t = b_2/\text{se}(b_2) \sim t_{(N-2)}$ if the null hypothesis is true
- Select $\alpha = 0.05$
  – The critical value for the right-tail rejection region is the 95th percentile of the *t*-distribution with $N - 2 = 38$ degrees of freedom, $t_{(0.95,38)} = 1.686$.
  – Thus we will reject the null hypothesis if the calculated value of $t \geq 1.686$.
  – If $t < 1.686$, we will not reject the null hypothesis.

- Using the food expenditure data, we found that $b_2 = 10.21$ with standard error $\text{se}(b_2) = 2.09$
  - The value of the test statistic is:

$$t = \frac{b_2}{\text{se}(b_2)} = \frac{10.21}{2.09} = 4.88$$

- Since $t = 4.88 > 1.686$, we reject the null hypothesis that $\beta_2 = 0$ and accept the alternative that $\beta_2 > 0$
  - That is, we reject the hypothesis that there is no relationship between income and food expenditure, and conclude that there is a *statistically significant* positive relationship between household income and food expenditure

- The null hypothesis is $H_0: \beta_2 \leq 5.5$

  The alternative hypothesis is $H_1: \beta_2 > 5.5$
- The test statistic is $t = (b_2 - 5.5)/\text{se}(b_2) \sim t_{(N-2)}$ if the null hypothesis is true
- Select $\alpha = 0.01$
  - The critical value for the right-tail rejection region is the 99th percentile of the *t*-distribution with $N - 2 = 38$ degrees of freedom, $t_{(0.99,38)} = 2.429$
  - Thus we will reject the null hypothesis if the calculated value of $t \geq 2.429$
  - If $t < 2.429$, we will not reject the null hypothesis

- Using the food expenditure data, we found that b_2 = 10.21 with standard error se(b_2) = 2.09
  - The value of the test statistic is:

$$t = \frac{b_2 - 5.5}{\text{se}(b_2)} = \frac{10.21 - 5.5}{2.09} = 2.25$$

- Since $t = 2.25 < 2.429$ we do not reject the null hypothesis that $\beta_2 \leq 5.5$
  - We are *not* able to conclude that the new supermarket will be profitable and will not begin construction

- The null hypothesis is $H_0\!:\!\beta_2 \geq 15$

  The alternative hypothesis is $H_1\!:\!\beta_2 < 15$
- The test statistic is $t = (b_2 - 15)/\text{se}(b_2) \sim t_{(N-2)}$ if the null hypothesis is true
- Select $\alpha = 0.05$
  - The critical value for the left-tail rejection region is the 5th percentile of the t-distribution with $N - 2 = 38$ degrees of freedom, $t_{(0.05,38)} = -1.686$.
  - Thus we will reject the null hypothesis if the calculated value of $t \leq -1.686$
  - If $t > -1.686$, we will not reject the null hypothesis

- Using the food expenditure data, we found that $b_2 = 10.21$ with standard error $\text{se}(b_2) = 2.09$
  - The value of the test statistic is:

$$t = \frac{b_2 - 15}{\text{se}(b_2)} = \frac{10.21 - 15}{2.09} = -2.29$$

- Since $t = -2.29 < -1.686$ we reject the null hypothesis that $\beta_2 \geq 15$ and accept the alternative that $\beta_2 < 15$
  - We conclude that households spend less than \$15 from each additional \$100 income on food

- The null hypothesis is $H_0\!:\beta_2 = 7.5$
  The alternative hypothesis is $H_1\!:\beta_2 \neq 7.5$
- The test statistic is $t = (b_2 - 7.5)/\text{se}(b_2) \sim t_{(N-2)}$ if the null hypothesis is true
- Select $\alpha = 0.05$
  - The critical value for the two-tail rejection region is the 2.5th percentile of the *t*-distribution with $N - 2 = 38$ degrees of freedom, $t_{(0.025,38)} = -2.024$ and the 97.5th percentile $t_{(0.975,38)} = 2.024$
  - Thus we will reject the null hypothesis if the calculated value of $t \geq 2.024$ **or** if $t \leq -2.024$

- Using the food expenditure data, we found that $b_2 = 10.21$ with standard error $\text{se}(b_2) = 2.09$
  - The value of the test statistic is

$$t = \frac{b_2 - 7.5}{\text{se}(b_2)} = \frac{10.21 - 7.5}{2.09} = 1.29$$

- Since $-2.024 < t = 1.29 < 2.024$ we do not reject the null hypothesis that $\beta_2 = 7.5$
  - The sample data are consistent with the conjecture households will spend an additional \$7.50 per additional \$100 income on food.

- The null hypothesis is $H_0{:}\beta_2 = 0$
  The alternative hypothesis is $H_1{:}\beta_2 \neq 0$
- The test statistic is $t = (b_2)/\text{se}(b_2) \sim t_{(N-2)}$ if the null hypothesis is true
- Select $\alpha = 0.05$
  - The critical value for the two-tail rejection region is the 2.5th percentile of the t-distribution with $N - 2 = 38$ degrees of freedom, $t_{(0.025,38)} = -2.024$ and the 97.5th percentile $t_{(0.975,38)} = 2.024$
  - Thus we will reject the null hypothesis if the calculated value of $t \geq 2.024$ **or** if $t \leq -2.024$

- Using the food expenditure data, we found that $b_2 = 10.21$ with standard error $\text{se}(b_2) = 2.09$
  - The value of the test statistic is

$$t = \frac{b_2}{\text{se}(b_2)} = \frac{10.21}{2.09} = 4.88$$

- Since 4.88 > 2.024 we reject the null hypothesis that $\beta_2 = 0$
  - We conclude that there is a statistically significant relationship between income and food expenditure

- From typical Eviews output, we can easily find the calculated *t* value used in this example

Variable	Coefficient	Std. Error	*t*-Statistic	Prob.
C	83.41600	43.41016	1.921578	0.0622
INCOME	10.20964	2.093264	4.877381	0.0000

3.5
The *p*-Value

- When reporting the outcome of statistical hypothesis tests, it has become standard practice to report the *p*-value (an abbreviation for probability value) of the test.
  - If we have the *p*-value of a test, *p*, we can determine the outcome of the test by comparing the *p*-value to the chosen level of significance, α, without looking up or calculating the critical values.
    - This is much more convenient

p-VALUE RULE

Reject the null hypothesis when the *p*-value is less than, or equal to, the level of significance α. That is, if $p \leq \alpha$ then reject H_0. If $p > \alpha$ then do not reject H_0.

- If t is the calculated value of the t-statistic, then:
  - if H_1: $\beta_K > c$

    p = probability to the right of t
  - if H_1: $\beta_K < c$

    p = probability to the left of t
  - if H_1: $\beta_K \neq c$

    p = <u>sum</u> of probabilities to the right of $|t|$

    <u>and</u> to the left of $-|t|$

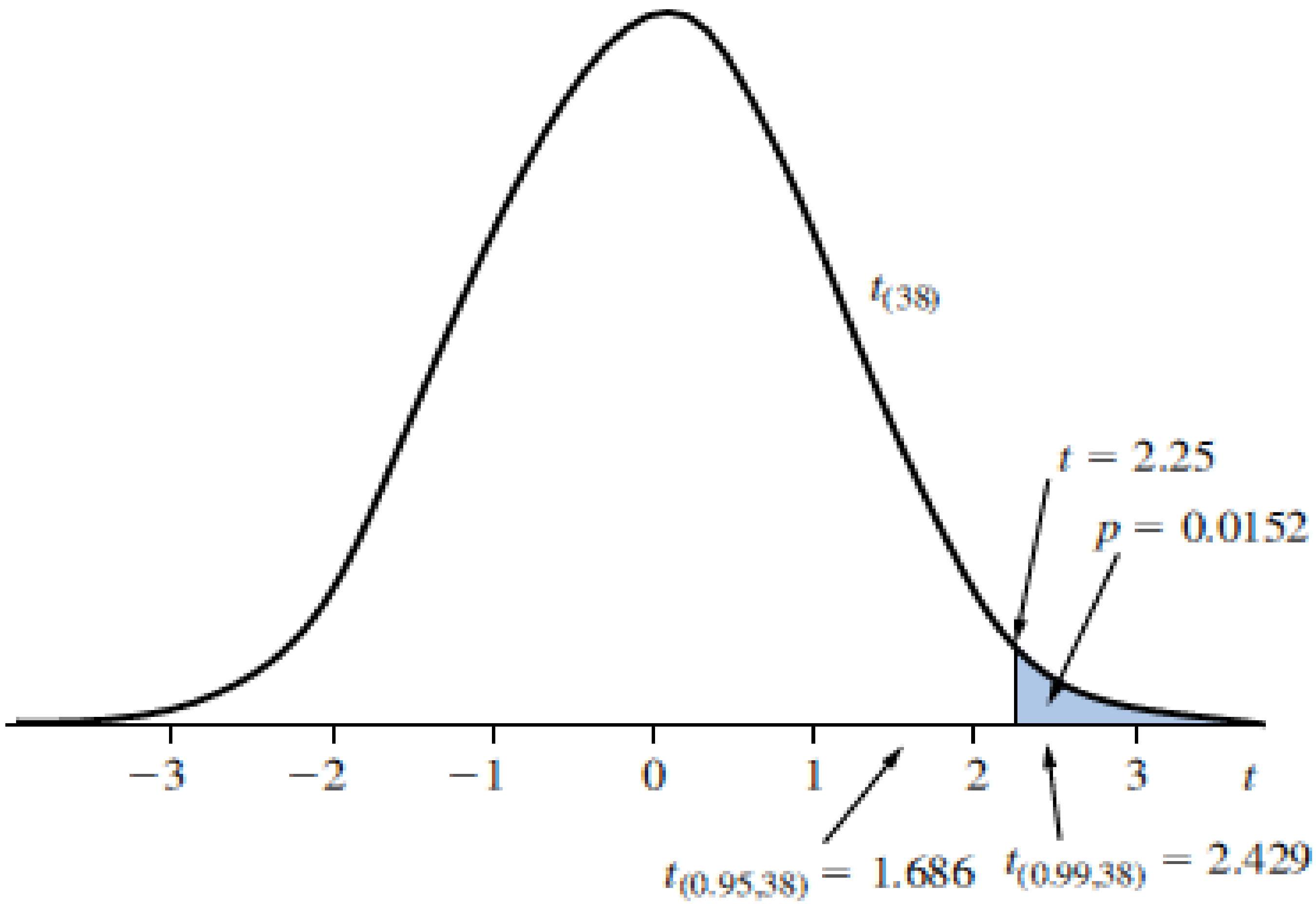
- From Section 3.4.1b, we have
  - The null hypothesis is H_0: $\beta_2 \leq 5.5$
  - The alternative hypothesis is H_1: $\beta_2 > 5.5$

$$t = \frac{b_2 - 5.5}{\text{se}\left(b_2\right)} = \frac{10.21 - 5.5}{2.09} = 2.25$$

  - The p-value is

$$p = P\left[t_{(38)} \geq 2.25\right] = 1 - P\left[t_{(38)} \leq 2.25\right] = 1 - 0.9848 = 0.0152$$

Figure 3.5 The *p*-value for a right-tail test.



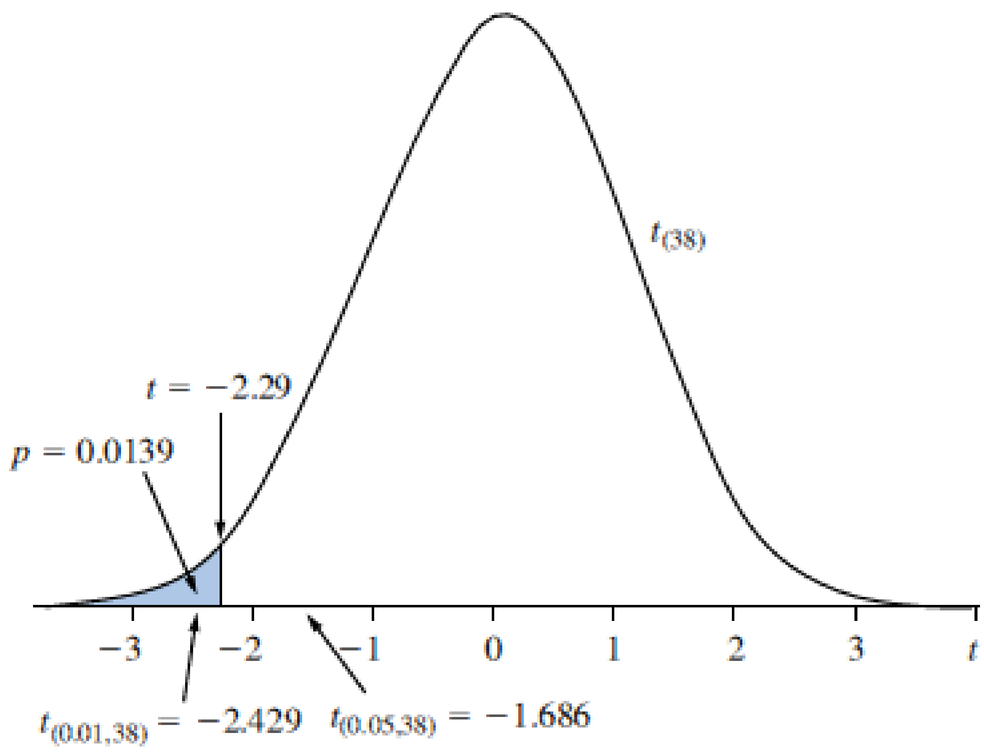
- From Section 3.4.2, we have
  - The null hypothesis is H_0: $\beta_2 \geq 15$
  - The alternative hypothesis is H_1: $\beta_2 < 15$

$$t = \frac{b_2 - 15}{\text{se}(b_2)} = \frac{10.21 - 15}{2.09} = -2.29$$

  - The p-value is

$$p = P\left[t_{(38)} \leq -2.29\right] = 0.0139$$

Figure 3.6 The *p*-value for a left-tail test.



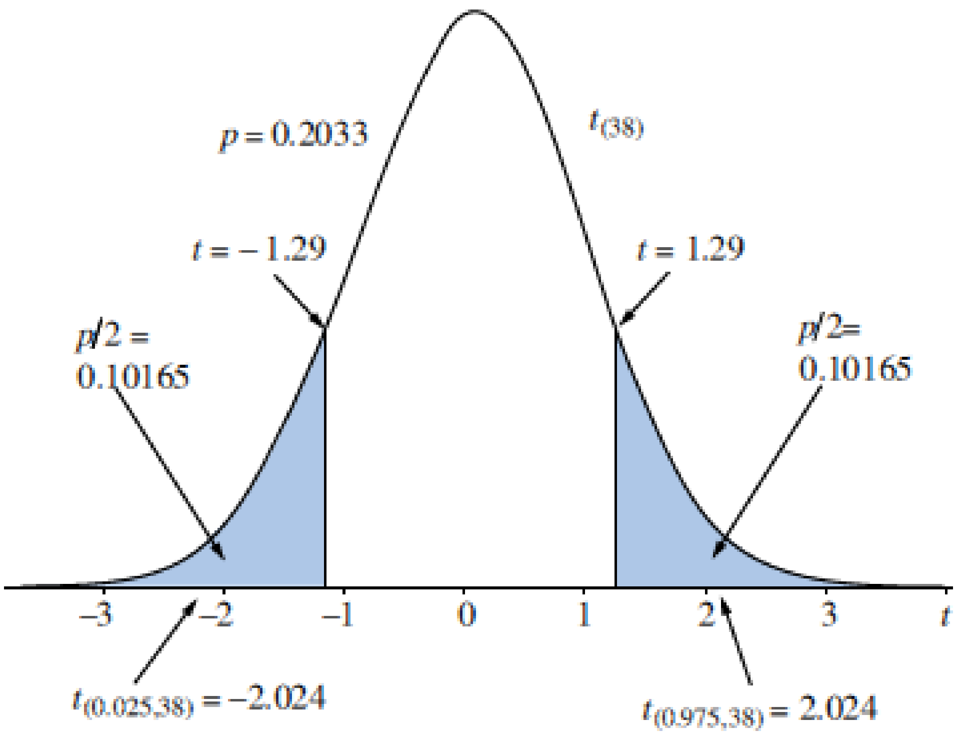
- From Section 3.4.3a, we have
  - The null hypothesis is H_0: $\beta_2 = 7.5$
  - The alternative hypothesis is H_1: $\beta_2 \neq 7.5$

$$t = \frac{b_2 - 7.5}{\text{se}(b_2)} = \frac{10.21 - 7.5}{2.09} = 1.29$$

  - The p-value is

$$p = P\left[t_{(38)} \geq 1.29\right] + P\left[t_{(38)} \leq -1.29\right] = 0.2033$$

Figure 3.7 The *p*-value for a two-tail test of significance.



- From Section 3.4.3b, we have
  - The null hypothesis is H_0: $\beta_2 = 0$
  - The alternative hypothesis is H_1: $\beta_2 \neq 0$

$$t = \frac{b_2}{\text{se}(b_2)} = \frac{10.21}{2.09} = 4.88$$

  - The p-value is

$$p = P\left[t_{(38)} \geq 4.88\right] + P\left[t_{(38)} \leq -4.88\right] = 0.0000$$

- From typical Eviews output, we can easily find the calculated *p*-value used in this example

Variable	Coefficient	Std. Error	*t*-Statistic	Prob.
C	83.41600	43.41016	1.921578	0.0622
INCOME	10.20964	2.093264	4.877381	0.0000

3.6
Linear Combinations of Parameters

- We may wish to estimate and test hypotheses about a linear combination of parameters $\lambda = c_1\beta_1 + c_2\beta_2$, where c_1 and c_2 are constants that we specify
  - Under assumptions SR1–SR5 the least squares estimators b_1 and b_2 are the best linear unbiased estimators of β_1 and β_2
  - It is also true that $\hat{\lambda} = c_1b_1 + c_2b_2$ is the best linear unbiased estimator of $\lambda = c_1\beta_1 + c_2\beta_2$

- As an example of a linear combination, if we let $c_1 = 1$ and $c_2 = x_0$, then we have

$$\begin{aligned}\lambda &= c_1\beta_1 + c_2\beta_2 \\ &= \beta_1 + x_0\beta_2 \\ &= E\left(y \mid x = x_0\right)\end{aligned}$$

which is just out basic model

- The estimator $\hat{\lambda}$ is unbiased because

$$\begin{aligned} E\left(\hat{\lambda}\right) &= E\left(c_1 b_1 + c_2 b_2\right) \\ &= c_1 E\left(b_1\right) + c_2 E\left(b_2\right) \\ &= c_1 \beta_1 + c_2 \beta_2 \\ &= \lambda \end{aligned}$$

- The variance of $\hat{\lambda}$ is

Eq. 3.8

$$\operatorname{var}\left(\hat{\lambda}\right) = \operatorname{var}\left(c_1 b_1 + c_2 b_2\right)$$
$$= c_1^2 \operatorname{var}\left(b_1\right) + c_2^2 \operatorname{var}\left(b_2\right) + 2c_1 c_2 \operatorname{cov}\left(b_1, b_2\right)$$

where the variances and covariances are given in Eq. 2.20 - 2.22

- We estimate $\hat{\lambda}$ by replacing the unknown variances and covariances with their estimated variances and covariances in Eq. 2.20 - 2.22

Eq. 3.9

$$\widehat{\text{var}}\left(\hat{\lambda}\right) = \widehat{\text{var}}\left(c_1 b_1 + c_2 b_2\right)$$
$$= c_1^2 \widehat{\text{var}}\left(b_1\right) + c_2^2 \widehat{\text{var}}\left(b_2\right) + 2c_1 c_2 \widehat{\text{cov}}\left(b_1, b_2\right)$$

- The standard error of $\hat{\lambda}$ is the square root of the estimated variance

Eq. 3.10

$$\operatorname{se}\left(\hat{\lambda}\right) = \operatorname{se}\left(c_1 b_1 + c_2 b_2\right) = \sqrt{\widehat{\operatorname{var}}\left(c_1 b_1 + c_2 b_2\right)}$$

- If in addition SR6 holds, or if the sample is large, the least squares estimators b_1 and b_2 have normal distributions.
  - It is also true that linear combinations of normally distributed variables are normally distributed, so that

$$\hat{\lambda} = c_1 b_1 + c_2 b_2 \sim N\left[\lambda, \operatorname{var}\left(\hat{\lambda}\right)\right]$$

- We can estimate the average (or expected) expenditure on food as:

$$\widehat{FOOD\_EXP} = 83.4160 + 10.2096\, INCOME$$

  – If the household income is \$2000, which is 20 since income is measured in \$100 units in this example, then the average expenditure is:

$$\begin{aligned} E\left(\widehat{FOOD\_EXP} \mid INCOME = 20\right) &= b_1 + b_2 20 \\ &= 83.4160 + 10.2096(20) \\ &= 287.6089 \end{aligned}$$

  – We estimate that the expected food expenditure by a household with \$2,000 income is \$287.61 per week

3.6.2
An Interval Estimate of Expected Food Expenditure

- The t-statistic for the linear combination is:

Eq. 3.11

$$t = \frac{\hat{\lambda} - \lambda}{\sqrt{\widehat{\text{var}}\left(\hat{\lambda}\right)}} = \frac{\hat{\lambda} - \lambda}{\text{se}\left(\hat{\lambda}\right)} = \frac{\left(c_1 b_1 + c_2 b_2\right) - \left(c_1 \beta_1 + c_2 \beta_2\right)}{\text{se}\left(c_1 b_1 + c_2 b_2\right)} \sim t_{(N-2)}$$

- Substituting the t value into $P(-t_c \le t \le t_c) = 1 - \alpha$, we get:

$$P\begin{bmatrix} (c_1 b_1 + c_2 b_2) - t_c \text{se}(c_1 b_1 + c_2 b_2) \le c_1\beta_1 + c_2\beta_2 \le \\ (c_1 b_1 + c_2 b_2) + t_c \text{se}(c_1 b_1 + c_2 b_2) \end{bmatrix} = 1 - \alpha$$

so that the $(1 - \alpha)\%$ interval is

$$(c_1 b_1 + c_2 b_2) \pm t_c \text{se}(c_1 b_1 + c_2 b_2)$$

- For our example, the estimated variances and covariance is:

	C	Income
C	1884.442	-85.9032
Income	-85.9032	4.3818

- The estimated variance of our expected food expenditure is:

$$\widehat{\text{var}}\left(b_1 + 20b_2\right) = \widehat{\text{var}}\left(b_1\right) + 20^2\,\widehat{\text{var}}\left(b_2\right) + 2\times 20\times \widehat{\text{cov}}\left(b_1, b_2\right)$$

$$= 1884.442 + 20^2 \times 4.3818 + 2\times 20\times\left(-85.9032\right)$$

$$= 201.0169$$

and the corresponding standard error is:

$$\text{se}\left(b_1 + 20b_2\right) = \sqrt{\widehat{\text{var}}\left(b_1 + 20b_2\right)}$$

$$= \sqrt{201.0169}$$

$$= 14.1780$$

- The 95% interval is then:

$$\left(\beta_1 + \beta_2 20\right) \pm t_{(0.975,38)} \mathrm{se}\left(b_1 + 20b_2\right)$$

or

$$\left[287.6089 - 2.024(14.1789), 287.6089 + 2.024(14.1789)\right]$$
$$= \left[258.91, 316.31\right]$$

  – We estimate with 95% confidence that the expected food expenditure by a household with$2,000 income is between $258.91 and $316.31

- A general linear hypothesis involves both parameters, β_1 and β_2 and may be stated as:

Eq. 3.12a

$$H_0 : (c_1\beta_1 + c_2\beta_2) = c_0$$

or, equivalently,

Eq. 3.12b

$$H_0 : (c_1\beta_1 + c_2\beta_2) - c_0 = 0$$

- The alternative hypothesis might be any one of the following:

$$(i)\quad H_1: c_1\beta_1 + c_2\beta_2 \neq c_0 \quad \text{two-tail test}$$

$$(ii)\quad H_1: c_1\beta_1 + c_2\beta_2 > c_0 \quad \text{right-tail test}$$

$$(iii)\quad H_1: c_1\beta_1 + c_2\beta_2 < c_0 \quad \text{left-tail test}$$

- The t-statistic is:

Eq. 3.13

$$t = \frac{\left(c_1\beta_1 + c_2\beta_2\right) - c_0}{\mathrm{se}\left(c_1\beta_1 + c_2\beta_2\right)} \sim t_{(N-2)}$$

if the null hypothesis is true

– The rejection regions for the one- and two-tail alternatives (i) – (iii) are the same as those described in Section 3.3, and conclusions are interpreted the same way as well

- Suppose we conjecture that:

$$E(FOOD\_EXP \mid INCOME = 20) = \beta_1 + \beta_2 20 > 250$$

- Use this as the alternative hypothesis:

$$H_1 : \beta_1 + \beta_2 20 > 250$$

or

$$H_1 : \beta_1 + \beta_2 20 - 250 > 0$$

- The null hypothesis is the logical alternative:

$$H_0 : \beta_1 + \beta_2 20 \leq 250$$

or

$$H_0 : \beta_1 + \beta_2 20 - 250 \leq 0$$

  - The null and alternative hypothesis are in the same form as the general linear hypothesis with $c_1 = 1$, $c_2 = 20$, and $c_0 = 250$

- The t-statistic is

$$t = \frac{\left(b_1 + 20b_2\right) - 250}{\text{se}\left(b_1 + 20b_2\right)}$$

$$= \frac{\left(83.4160 + 20 \times 10.2096\right) - 250}{14.1780}$$

$$= \frac{287.6089 - 250}{14.1780}$$

$$= \frac{37.6089}{14.1780}$$

$$= 2.65$$

- Since $t = 2.65 > t_c = 1.686$, we reject the null hypothesis that a household with weekly income of $2,000 will spend $250 per week or less on food, and conclude that the conjecture that such households spend more than $250 is correct, with the probability of Type I error 0.05

Key Words

Keywords

- alternative hypothesis
- confidence intervals
- critical value
- degrees of freedom
- hypotheses
- hypothesis testing
- Inference
- interval estimation
- level of significance
- linear hypothesis
- null hypothesis
- one-tail tests
- point estimates
- probability value
- *p*-value
- rejection region
- test of significance
- test statistic
- two-tail tests
- Type I error
- Type II error

Appendices

- 3A Derivation of the t-distribution
- 3B Distribution of the t-statistic under H_1
- 3C Monte Carlo Simulation

- Consider the normal distribution of b_2, the least squares estimator of β_2, which we denote as:

$$b_2 \sim N\left(\beta_2, \frac{\sigma^2}{\sum\left(x_i - \bar{x}\right)^2}\right)$$

- The standardized normal is:

Eq. 3A.1

$$Z = \frac{b_2 - \beta_2}{\sqrt{\operatorname{var}\left(b_2\right)}} \sim N\left(0,1\right)$$

- If all the random errors are independent, then

Eq. 3A.2

$$\sum\left(\frac{e_i}{\sigma}\right)^2=\left(\frac{e_1}{\sigma}\right)^2+\left(\frac{e_2}{\sigma}\right)^2+\ldots+\left(\frac{e_N}{\sigma}\right)^2\sim\chi^2_{(N)}$$

- Since the true random errors are unobservable, we replace them by their sample counterparts, the least squares residuals to obtain:

Eq. 3A.3

$$V=\frac{\sum\hat{e}_i^2}{\sigma^2}=\frac{(N-2)\hat{\sigma}^2}{\sigma^2}$$

- Therefore,

Eq. 3A.4

$$V = \frac{(N-2)\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{(N-2)}$$

  – Although we have not established the fact that the chi-square random variable V is statistically independent of the least squares estimators b_1 and b_2, it is
  – Consequently, V and the standard normal random variable Z in Eq. 3A.1 are independent

- A t-random variable is formed by dividing a standard normal random variable, $Z \sim N(0, 1)$, by the square root of an independent chi-square random variable, $V \sim \chi^2_{(m)}$, that has been divided by its degrees of freedom, m
  - That is:

$$t = \frac{z}{\sqrt{V/m}} \sim t_{(m)}$$

- Using Z and V from Eq. 3A.1 and Eq. 3A.4, respectively, we have:

Eq. 3A.5

$$t = \frac{Z}{\sqrt{V/(N-2)}}$$

$$= \frac{(b_2 - \beta_2)\Big/\sqrt{\sigma^2 / \sum (x_i - \bar{x})^2}}{\sqrt{\dfrac{(N-2)\hat{\sigma}^2/\sigma^2}{N-2}}}$$

$$= \frac{(b_2 - \beta_2)}{\sqrt{\dfrac{\hat{\sigma}^2}{\sum (x_i - \bar{x})^2}}} = \frac{(b_2 - \beta_2)}{\sqrt{\widehat{\operatorname{var}}(b_2)}} = \frac{(b_2 - \beta_2)}{\operatorname{se}(b_2)} \sim t_{(N-2)}$$

- To examine the distribution of the *t*-statistic in Eq. 3.7 when the null hypothesis is not true, suppose that the true $\beta_2 = 1$
  - We can show that:

$$t = \frac{(b_2 - 1)}{se(b_2)} \sim t_{(N-2)}$$

Eq. 3B.1

- If $\beta_2 = 1$ and we incorrectly hypothesize that $\beta_2 = c$, then the numerator in Eq. 3A.5 that is used in forming Eq. 3.7 has the distribution:

$$\frac{b_2 - c}{\sqrt{\operatorname{var}(b_2)}} \sim N\left(\frac{1-c}{\sqrt{\operatorname{var}(b_2)}}, 1\right)$$

  – Since its mean is not zero, the distribution of the variable in Eq. 3B.1 is not standard normal, as required in the formation of a t-random variable

- When studying the performance of hypothesis tests and interval estimators it is necessary to use enough Monte Carlo samples so that the percentages involved are estimated precisely enough to be useful
  - For tests with probability of Type I error $\alpha = 0.05$ we should observe true null hypotheses being rejected 5% of the time
  - For 95% interval estimators we should observe that 95% of the interval estimates contain the true parameter values
  - We use $M = 10{,}000$ Monte Carlo samples so that the experimental error is very small

Table 3C.1 Results of 10000 Monte Carlo Simulations

SAMPLE	*B2*	*SE*	*TSTAT*	*REJECT*	*LB*	*UB*	*COVER*
101	8.3181	1.5024	−1.1195	0	5.2767	11.3595	1
102	10.9564	1.5488	0.6175	0	7.8210	14.0918	1
103	13.3644	1.7085	1.9692	1	9.9057	16.8230	1
104	9.7406	1.8761	−0.1383	0	5.9425	13.5386	1
105	12.3402	1.6275	1.4379	0	9.0454	15.6350	1
106	11.9019	1.6031	1.1864	0	8.6567	15.1472	1
107	8.7278	1.2252	−1.0383	0	6.2475	11.2081	1
108	9.0732	1.6978	−0.5459	0	5.6361	12.5102	1
109	9.5502	1.4211	−0.3165	0	6.6734	12.4270	1
110	9.2007	1.4895	−0.5366	0	6.1854	12.2161	1
111	11.0090	1.5221	0.6629	0	7.9277	14.0903	1
112	12.7234	1.4783	1.8423	1	9.7308	15.7160	1
113	11.8995	1.7587	1.0801	0	8.3393	15.4597	1
114	12.9712	1.4679	2.0242	1	9.9997	15.9427	1
115	10.6347	1.6320	0.3889	0	7.3309	13.9385	1
116	10.0045	1.4179	0.0031	0	7.1341	12.8748	1
117	11.2658	1.5584	0.8123	0	8.1110	14.4206	1
118	11.4842	1.4449	1.0272	0	8.5592	14.4093	1
119	9.6915	1.7422	−0.1771	0	6.1647	13.2183	1
120	11.6990	1.5132	1.1228	0	8.6358	14.7623	1

- The lesson is, that in many repeated samples from the data generation process, and if assumptions SR1–SR6 hold, the procedure for constructing 95% interval estimates "works" 95% of the time

- The lesson is that in many repeated samples from the data generation process, and if assumptions SR1–SR6 hold, the procedure for testing a true null hypothesis at significance level $\alpha = 0.05$ rejects the true null hypothesis 5% of the time
  - Or, stated positively, the test procedure does not reject the true null hypothesis 95% of the time

- The point is that if fewer Monte Carlo samples are chosen the “noise” in the Monte Carlo experiment can lead to a percent of successes or rejections that has too wide a margin of error for us to tell whether the statistical procedure, interval estimation, or hypothesis testing, is “working” properly or not